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Patterns and generalization: the influence of visual strategies

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This paper gives a description of an ongoing study focused on pattern exploration and generalization tasks and justifies that study with respect to the literature. It promises to make a contribution to appreciation of the ways in which visual strategies can be used to enhance and enrich learners' experience of generalization. The main purpose is to analyse the strategies and difficulties presented by grade 6 students when solving these activities, along with the role played by visualization in their reasoning. Preliminary results indicate that, in general, students prefer analytic approaches over visual ones and that, among the group of students that are most successful, the majority chooses a mixed strategy.

THEORETICAL FRAMEWORK

A quarter of a century ago, problem solving became a focus of school mathematics. According to recent curricular guidelines of several countries, one of the main purposes of mathematics learning is the development of the ability to solve problems. In spite of the growing curricular relevance of this theme over the last few years, several international studies (SIAEP, TIMSS, PISA) have shown that Portuguese students have low results when solving problems is required (Ramalho, 1994; Amaro, Cardoso & Reis, 1994; OECD, 2004). Pattern exploration tasks may contribute to the development of abilities related to problem solving, through emphasising the analysis of particular cases, organizing data systematically, conjecturing and generalizing. The *Principles and Standards for School Mathematics* (NCTM, 2000) acknowledges the importance of working with numeric, geometric and pictorial patterns. This document states that instructional mathematics programs should enable students, from pre-kindergarten through grade 12, to engage in activities involving the understanding of patterns, relations and functions. On the other hand, Geometry is considered a source of interesting problems that can help students develop abilities such as visualization, reasoning and mathematical argumentation. Visualization, in particular, is an important part of mathematical reasoning but, according to some studies, its role hasn't always been emphasized in students' mathematical experiences (Hadamard, 1973). According to Dreyfus (1991) visual reasoning in mathematics is important in itself. Therefore it's necessary to give increased status to purely visual mathematical tasks. The usefulness of visualization and graphical representations is being recognized by many mathematics educators. However more research is still necessary concerning the role mental images play in the understanding of mathematical concepts and in problem solving. Research is also needed to ascertain when visualization is more useful than analytical methods (Gutiérrez, 1996).

The purpose of this study is to analyse the difficulties and strategies used by grade 6 students (11-12 years old), when solving problems involving pattern search, and the role played by visualization in their reasoning. The tasks that will be used in the study involve pattern generalization. Students of this age have not yet had formal algebra instruction, thus the importance of analysing their approaches. This study attempts to address the following research questions:

1) Which difficulties do 6th grade students present when solving pattern exploration tasks?

2) How can we characterize students' strategies?

3) What's the role played by visualization in the mediation of students' reasoning? Will it simplify the path to solution or will it act as a blocking element?

Patterns and generalization

Using patterns to promote and provoke generalization is seen by many as a pre-algebraic activity (e.g. Mason et al., 1985; Mason, 1996; Lee, 1996). The focus on pattern exploration is frequent in the recent approaches to the study of algebra. The search for regularities in different contexts, the use of symbols and variables that represent patterns and generalization are important components of the math curriculum in many countries. Portuguese curriculum recommends that students should develop the predisposition to search and explore number and geometric patterns throughout elementary and middle school (DEB, 2001).

Research on students' thinking processes in generalization

There are now several studies about the analysis and development of pattern-finding strategies with students from pre-kindergarten to secondary school.

Stacey (1989) focused her investigation on the generalization of linear patterns, with students aged 9-13 years old. A significant number of the subjects used an erroneous direct proportion method in an attempt to generalize. Stacey also reported some inconsistencies in the strategies used by students in *near generalization* (activities that can be solved by the use of a drawing or the recursive method) and *far generalization* (the strategies stated before are not adequate to these kind of activities, they imply the finding of a rule) and concluded that drawing had a major influence on their approaches, although she didn't explore this theme further.

García Cruz & Martinón (1997) developed a study, with 15-16 years old students, aiming to analyse the way they validate results and to ascertain if they favoured numerical or geometric strategies. The research showed that drawing played a double role on the process of abstracting and generalizing. It represented the setting for students who used visual strategies in order to achieve generalization and, on the other hand, acted as a means to check the validity of the reasoning for students who favoured numerical strategies.

Orton & Orton (1999) focused their investigation on linear and quadratic patterns with 10-13 years old students. They reported a tendency to use differences between consecutive elements and its extension to quadratic patterns, by taking second differences, but without success in some cases. They also pointed to students' arithmetical incompetence and their fixation on a recursive approach as some of the obstacles to successful generalization.

Sasman et al. (1999) developed a study with 8th grade students, involving generalization tasks with variation of the representations. Results showed that students used, almost exclusively, number context, neglecting drawings, and favoured the recursive method, making several mistakes related to the erroneous use of direct proportion.

Mason et al. (2005) promotes the use of the strategy of 'Watch What You Do' as learners draw further cases of patterns and attend to how they naturally draw the pattern efficiently. Each such efficient drawing method offers a potential generalization when expressed as instructions as to how to draw the pattern.

Visualization

The relation between the use of visual abilities and students' mathematical performance constitutes an interesting area for research and does not achieve consensus. Many researchers recognize the importance of the role that visualization plays in problem solving, while others claim that visualization alone isn't enough, that it must be used as a complement to analytic reasoning. According to Presmeg (1986), teachers have a tendency to present visual reasoning only as a possible strategy for problem solving in an initial stage or as a complement to analytic methods.

Thornton (2001) points to three reasons to re-evaluate the role of visualization in school mathematics: (1) math is currently identified with the study of patterns and that, together with the use of technology, has the power to demean the difficulty of algebraic thinking; (2) visualization can often provide simple and powerful approaches to problem solving; (3) teachers should recognize the importance of helping students develop a repertoire of techniques to approach mathematical situations.

Different students can use different strategies when solving the same problem. Some prefer visual methods, others are in favour of non-visual ones. Krutetskii's (1976) study with mathematically gifted students showed that they use different approaches to problems, leading to the following categorization concerning reasoning: analytic (non- visual), geometric (visual) and harmonic (use of the two previous types of reasoning).

In spite of the preference for the use of numerical relations as a support for reasoning, in part due to the work promoted in the classroom, some studies indicate that most are more successful when they use a harmonic or mixed approach (Moses, 1982; Noss, Healy & Hoyles, 1997; Stacey, 1989).

METHOD

The sample used on this study consists of three classes of grade 6 students, from three different schools, aged 11-12 years, corresponding to a total of 54 students. The study is divided in three stages: the first corresponds to the administration of a test that focuses on pattern exploration and generalization problems; the second stage involves the implementation of tasks, of the same nature, to all students, in pairs; on the third, students will repeat the test in order to examine changes in the results. The second stage of the study is, at the present time, in the beginning. All students will be involved in solving 10 tasks over the school year and two pairs of students from each school will be selected for clinical interviews. These sessions will be videotaped for further analysis in order to investigate students' mathematical reasoning, in particular the strategies used to solve each of the problems posed, as well as the difficulties they experienced on that activity.

There will be qualitative and quantitative data. To gather the quantitative data a scoring scale for the test was developed in order to compare the two applications. Qualitative data will be collected from the interviews with the elements of the pairs and from the analysis of the strategies used in the tests.

PRELIMINARY RESULTS

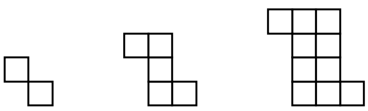
At present time only the first stage of the study is concluded. Students were given a written test with pre-algebraic questions. The test contains sixteen introductory questions consisting of visual and numerical sequences **(see a)**, for an example**)**, followed by two more complex tasks involving *near* and *far generalization* **(see b) and c))**.

a) Examples of introductory questions:

1. *Complete the following sequences indicating the next **two** elements:*

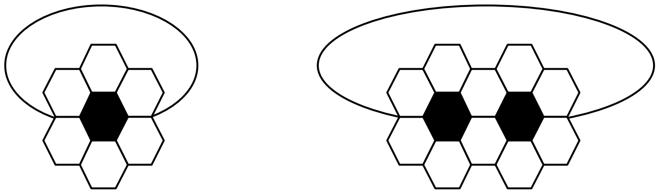
   1.2: 2, 5, 8, 11, 14

   1.13:



b) Second task:

2. *Joana likes to make necklaces using flowers. She uses white beads for the petals and black beads for the centre of each flower. The figure below shows a necklace with one flower and a necklace with two flowers, both made by her.*



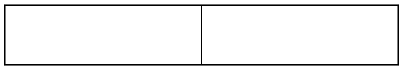
2.1. *How many white and black beads will Joana need to make a necklace with 3 flowers?* ***Explain your conclusion****.*

2.2. *How many white and black beads will Joana need to make a necklace with 8 flowers?* ***Explain your conclusion****.*

2.3. *If Joana wants to make a necklace with 25 flowers, how many white and black beads will she need?* ***Explain your conclusion****.*

c) Third task:

3. *On the following figure you can count three rectangles.*



Consider the figure below:

3.1. *How many rectangles of different sizes can you find?* ***Explain your reasoning****.*

3.2. *If you had 10 rectangles in a row, how many rectangles of different sizes could you count?* ***Explain your reasoning****.*

The test was constructed with the purpose of analysing students' abilities when performing pattern seeking and generalization tasks and of studying their problem solving approaches. It was validated by a panel of teachers and researchers in mathematics education. It was also solved by 5th and 6th grade students of different schools before its implementation.

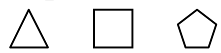
The application of this test with the sample used in this study made it possible to collect some preliminary data about students' thinking processes and most common difficulties.

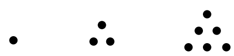
Thinking strategies that emerged from the application of the test

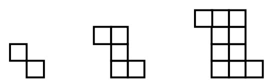
In spite of being given an image of the first two elements of the sequence, in the second task students rarely use drawing as a solving strategy, they favoured a numerical approach. Some students made a drawing to solve the first two questions and applied direct counting to determine the number of beads, but they weren't able to solve the last question by the same method, since it involved *far generalization*, so they left it in blank or presented a feeble attempt to solve it. The few students that have successfully solved this task used a mixed strategy, presenting numeric relations and referring to the visual structure of the sequence.

Third task was considered by the students as the most difficult. No one could reach a solution. Some students identified the existence of different rectangles but, as they didn't found an organized way to approach the question, they couldn't find all the cases. In the second question of this task no figure was given. Most students started by representing the situation, but in the end they weren't able to discover the pattern due to the application of inadequate strategies like direct counting or the use of a confusing diagram.

Difficulties emerging from the application of the test

The greatest facility was achieved on the first task of the test, possibly because they had prior experience solving this type of tasks. Nevertheless they showed some difficulties that should be pointed. Some of the sequences were interpreted, by several students, as repetition patterns, both on visual and numerical contexts. The two most frequent cases happened with the numerical sequence *1, 4, 9, 16* and the visual  sequence . Students continued the first by adding 3 to 16 and 5 to 19, instead of continuing the sequence of squares of whole numbers. In the second case, we expected to get a hexagon and a heptagon and some students presented a triangle and a square, repeating the sequence. The majority of students achieved better results on the questions involving numerical patterns than on those involving visual patterns. They presented very low scores in completing the following two sequences, whose nature was purely visual:





In our opinion, the first one caused some difficulties possibly due to the triangular shape of the elements of the sequence and the second one due to the simultaneous variation of length and height.

On the second task there's a general tendency to the erroneous use of the direct proportion method. This indicates that students didn't analyse properly the structure of the sequence, thinking of each flower as a disjoint unit. Most of them considered that each flower had six white beads and one black, so a necklace with eight flowers would have forty-eight white beads and eight black and a necklace with twenty five flowers would have hundred and fifty white beads and twenty five black. These students didn't notice that consecutive flowers had two white beads in common. They would easily see the error if they checked the rule with a drawing. On the last two questions of this task, which involve the use of the recursive method or the finding of a general rule, scores were very low. In our opinion, students' tendency to manipulate numbers only, may have contributed to enhance the difficulty of finding the pattern in question.

The last task of the test was the most difficult for students to solve. The majority identified only the smaller rectangles and the bigger one, possibly influenced by the example given in the problem. In some cases, they used the direct proportion to determine the number of rectangles, similarly to the previous task, considering that if they had ten rectangles in a row, then they would have to duplicate the result obtained in the first question.

DISCUSSION

Work with patterns may be considered a unifying theme of mathematics teaching, appearing in different contexts and contributing to the development of several concepts (NCTM, 2000). In this research, the use of pattern exploration tasks has the main purpose of setting the environment to analyse the impact of the use of visual strategies in generalization.

Some studies indicate that students prefer analytic approaches to mathematical activities, converting into numbers even problems that have a visual nature. At this point this research is one more contribution to this view. Research on visualization and on the role of mental images in mathematical reasoning has shown the importance of representations in conceptual development (Palarea & Socas, 1998). Our expectations, at this moment, are that, on the second stage of the study, with the implementation of the tasks, students will then use more frequently visual or mixed strategies and develop a higher competence in solving pre-algebraic activities.

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