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DEVELOPING STUDENTS' FLEXIBILITY ON PATTERN GENERALIZATION

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Abstract

This paper refers to a study developed with fifty-four 6th grade students. The main goal was to analyse their performance when solving tasks involving the generalization of visual patterns. In order to better understand this problem we focussed on the following features: type of generalization strategies used; difficulties that emerged from students' work; and the role played by visualization on their reasoning. One of the main purposes of mathematics education is to promote flexibility in exploring mathematical ideas, through the use and combination of different methods to solve problems. In this sense, in this paper we will present results related to the implementation of some of the tasks.

Keywords: *problem solving, patterns, generalization strategies, flexibility.*

Introduction

The 80's are an important landmark for school mathematics. In this decade deep curricular changes were made and problem solving became a fundamental part of all mathematics learning (NCTM, 2000). This idea is still current in the recent curricular guidelines of several countries, where the ability to solve problems is mentioned as one of the main goals of learning mathematics. In spite of the relevance given to this subject, some international studies (SIAEP, TIMSS, PISA) have shown that Portuguese students perform badly when solving problems (Ramalho, 1994; Amaro, Cardoso & Reis, 1994; OCDE, 2004). However, there is a common thought that pattern exploration tasks may contribute to the development of abilities related to problem solving, through emphasising the analysis of particular cases, organizing data in a systematic way, conjecturing and generalizing. The *Principles and Standards for School Mathematics* (NCTM, 2000) acknowledges the importance of working with numeric, geometric and pictorial patterns stating that instructional mathematics programs should enable students, from pre-kindergarten to grade 12, to engage in activities involving understanding patterns, relations and functions. Work with patterns may also be helpful in building a more positive and meaningful image of mathematics and contribute to the development of several skills, in particular related to problem solving and algebraic thinking (Vale et al, 2006). On the other hand, Geometry is considered a source of interesting problems that can help students develop abilities such as visualization, reasoning and argumentation, so important in problem solving. Visualization, in particular, is an essential mathematical ability but, according to some studies, its role hasn't always been emphasized in students' mathematical experiences (Healy & Hoyles, 1996; Presmeg, 2006). Although the usefulness of visualization is being recognized by many mathematics educators, in Portuguese classrooms teachers privilege numeric aspects over geometric ones. Considering it all, more research is still necessary regarding the role images play in the understanding of mathematical concepts and particularly in problem solving. This study intends to understand how 6^{th} grade students (11-12 years old) solve problems involving visual patterns. The tasks used in the study require pattern generalization and students of this age have not yet had formal algebra instruction, thus the importance of analysing the nature of their approaches. This study attempts to address the following research questions:

1) Which difficulties do 6^{th} grade students present when solving pattern exploration tasks?

2) How can we characterize students' generalization strategies?

3) What's the role played by visualization on students' reasoning?

Theoretical framework

Many mathematicians share an enthusiastic view about the role of patterns in mathematics, some even consider mathematics as being the science of patterns (Devlin, 2002; Steen, 1990). This perspective highlights the presence of patterns in all areas of mathematics, thus being a transversal and unifying theme. The mathematics curricula of many countries contemplate significant components related to patterns like: searching for patterns in different contexts; using and understanding symbols and variables that represent patterns; and generalizing. Portuguese curriculum mentions the importance of developing abilities like searching and exploring numeric and geometric patterns, as well as solving problems, looking for regularities, conjecturing and generalizing (DEB, 2001; ME-DGIDC, 2007).

There has been significant research about students' generalization strategies, from pre-kindergarten to secondary school. The adjustment of some frameworks proposed by different investigators (Lannin, 2003; Lannin, Barker & Townsend, 2006; Orton & Orton, 1999; Rivera & Becker, 2005; Stacey, 1989; Swafford & Langrall, 2000) led me to the following categorization (Barbosa, 2010):

Strategy		**Description**
Counting (C)		Drawing a figure and counting the desired elements.
Whole-object	No adjustment (W_1)	Considering a term of the sequence as unit and using multiples of that unit.
	Numeric adjustment (W_2)	Considering a term of the sequence as unit and using multiples of that unit. A final adjustment is made based on numeric properties.
	Visual adjustment (W_3)	Considering a term of the sequence as unit and using multiples of that unit. A final adjustment is made based on the context of the problem.
Difference	Recursive (D_1)	Extending the sequence using the common difference, building on previous terms.
	Rate - no adjustment (D_2)	Using the common difference as a multiplying factor without proceeding to a final adjustment.
	Rate - adjustment (D_3)	Using the common difference as a multiplying factor and proceeding to an adjustment of the result.
Explicit (E)		Discovering a rule, based on the context of the problem, that allows the immediate calculation of any output value given the correspondent input value.
Guess and check (GC)		Guessing a rule by trying multiple input values to check its' validity.

Table 1. Generalization Strategies Framework

These strategies often emerge through different types of reasoning and it's fundamental that students understand the potential and limitations of each approach. Depending on the type of task, some strategies may be more adequate than others and, on the other hand, can even lead students to difficulties or incorrect answers.

Patterning activities can be developed in a variety of contexts (numeric, geometric, concrete and visual) and through the use of different approaches. Gardner (1993) claims that some individuals recognize regularities spatially or visually, while others notice them logically or analytically. In fact, it's common, in mathematical activities, that different individuals process information in different ways. Many students favour analytic methods while others have a tendency to reason visually. The relation between the use of visual abilities and students' mathematical performance constitutes an interesting area for research. Many investigators stress the importance of the role visualization plays in problem solving (Presmeg, 2006; Shama & Dreyfus, 1994), while others claim that visualization should only be used as a complement to analytic reasoning (Goldenberg, 1996; Tall, 1991). In spite of some controversy, these visions reflect the importance of using and developing visual abilities in mathematics, not only analytic ones, but teachers tend to present visual reasoning only as a possible strategy for problem solving in an initial stage or, when necessary, as a complement to analytic methods (Presmeg, 1986). All students should benefit from learning situations where data is given and/or treated using parallel approaches, so that they become more able to use different strategies and chose the most adequate to solve a given problem (Mason, Johnston-Wilder & Graham, 2005). In this sense, teachers can contribute to the development of students' mathematical talent by motivating the flexibility to use strategies of different nature (Presmeg, 1986).

Method

Fifty four sixth-grade students (11-12 years old), from three different schools in the North of Portugal, participated in this study over the course of a school year. For six months all students involved in the study solved seven tasks, working in pairs, and two pairs from each school were selected for clinical interviews. There was a moment of discussion at the end of each cycle where different approaches to solve the same questions were analysed as well as the inadequacy of some strategies. The tasks applied along the study required near and far generalization and featured increasing and decreasing linear patterns as well as non linear ones. This paper reports some results from the application of two of the tasks.

Results

Generalization strategies

One of the selected tasks was called *Pins and Cards* (appendix 1). This was the first task solved by the students and represents an increasing linear pattern, presented visually.

Table 2 synthesizes the number of pairs of students that used a given strategy, based on the categories described on the *Generalization Strategies Framework* (table 1). In some cases it was impossible to categorize students' answers, those cases appear in the last column of the table, not categorized (*NC*). Trough this table is possible to analyze not only the approach used to solve each of the questions of the task, but also compare it with the level of generalization involved (near or far).

	C	**W_1**	**W_2**	**W_3**	**W**	**D_1**	**D_2**	**D_3**	**D**	**E**	**GC**	**NC**
1.	16	8	-	1	9	1	1	-	2	-	-	-
2.	-	3	2	1	6	3	1	-	4	12	-	5
3.	-	2	1	-	3	-	4	3	7	9	-	8

Table 2. Summary of the strategies used by the students

The first question of this task required near generalization. This type of questions can easily be solved by making a drawing of the requested term of the sequence and counting its elements, using the *counting* strategy. As we can see from table 2 *counting* over a drawing was the predominant strategy in near generalization, always leading to a correct answer. The *whole-object* strategy also emerged from the work of some of the pairs. This approach is associated to direct proportion situations and this particular problem does not fit this model. Nevertheless, eight pairs of students used proportional reasoning, duplicating the number of pins associated to the three cards. For this strategy to be adequate, students had to make a final adjustment based on the context. Only one of the pairs felt the need to adjust the result obtained in the duplication of the number of pins of the three cards. According to the literature (e. g. Orton & Orton, 1999; Stacey 1989) this type of tasks can promote the use of *recursive* thinking, especially when near generalization is involved. Curiously only one pair of students extended the sequence using the common difference to solve this question. Another case was registered in which the *difference* strategy was employed but in an incorrect way. To obtain the number of pins necessary to hang 6 cards, these students used a multiple of the common difference without adjusting the result, as happened in other cases with the *whole-object* strategy. The *explicit* and *guess and check* strategies were not applied to solve this question.

Although both questions 2 and 3 require far generalization, the third question of the task had a different structure, involving reverse thinking. When approaching far generalization students revealed more difficulties and that can be seen by the increasing number of not categorized answers, that represent imperceptible reasoning or no answer at all (table 2). We can notice in table 2 that students dropped the *counting* strategy when solving these two questions. Some pairs did start by using it but gave up along the way, claiming that "there were too many cards". Instead, the application of *explicit* strategies prevailed. Those who relied on this approach, using the context to identify an immediate relationship between the two variables, presented a high level of efficiency. Some students "saw" that each card needed three pins and the last one would need four, deducing that the rule was $3(n-1)+4$, n being the number of cards. Other pairs "saw" the pattern differently considering that each card had three pins adding one more pin at the end. Here the rule was $3n+1$. In fact, research on pattern and generalization shows that individuals might see the same pattern differently (Rivera & Becker, 2007), originating equivalent expressions. The *whole-object* strategy continued to appear as in the previous question, but this time a new approach emerged. Some students considered multiples of known terms of the sequence and adjusted the result based only on numeric properties. Students use proportional reasoning to determine the number of pins and, when adjusting the result they don't consider the context of the problem, only numeric properties, obtaining an incorrect answer. Comparing the first question with the last two, we can see that the use of the *difference* strategy increases. Some students gave up *counting*, as the order of the term became far, and started basing their reasoning on the common difference between terms. In the third question of the task, we can notice that three pairs of students applied a strategy that hasn't been used before. The difference between consecutive terms is of three pins, so, in this case, students used that fact to approach the number of pins available. Knowing the structure of the pattern, they were able to criticize the result, adjusting it.

The Sole Mio Pizzeria task (appendix 2) was solved four months later. The problem is similar to the one presented on the previous task, exhibiting an increasing linear pattern and contemplating near and far generalization. In order to compare the strategies, selected by students in this task, with the strategies used in the *previous* task, the categories were organized in the following table:

	C	W_1	W_2	W_3	W	D_1	D_2	D_3	D	E	GC	NC
1.	21	-	-	-	-	4	-	-	4	2	-	-
2.	1	-	-	-	-	3	-	1	4	22	-	-
3.	-	-	-	-	-	2	3	-	5	14	5	3

Table 3. Summary of the strategies used by the students

One of the most obvious facts is the lack of preference for the whole-object strategy. Being a linear pattern, the use of proportional reasoning is not adequate, unless an adjustment based on the context is made. It's possible that the fact that, in this case, the adjustment was more complex than in the previous problem could justify the absence of this approach. Counting is once again the privileged strategy in near generalization. It is applied by the majority of the students and this preference has increased compared to the previous task. Other strategies emerged but only a minority of students used them. Four pairs used recursive reasoning to extend the sequence to the 10th term and two pairs applied an explicit reasoning.

In the first task, explicit strategies only appeared when students were dealing with far generalization so it's surprising that they used it at this stage, showing that they immediately discovered the structure of the pattern. As in the Pins and Cards task, when dealing with far generalization, students do not recognize the usefulness of counting and that's why it has no expression on table 3, as they progress to far generalization. On the other hand, explicit reasoning prevails being implemented by even more students in a successful way. All of them described the pattern as 2n+2, n being the number of pizzas. They frequently referred that "in front of each pizza are two people and one more at each end of the table". Some students chose a safe path going with a recursive approach, through the extension of the sequence using the common difference. Similarly to what happened in the previous task, there were three pairs that considered multiples of the common difference but neglected to adjust the result, showing that their work was merely based on number relations. It's evident the use of a new strategy, guess and check, that was only applied in far generalization when reverse thinking was involved. Students identified the relation between the two variables and then tried some numbers until they achieved the wanted result.

Difficulties emerging from students work

When solving the first task some students struggled with cognitive difficulties that led to incorrect answers. Some pairs made false assumptions about the use of direct proportion. In these cases attention tended to focus only on numeric attributes with no appreciation of the structure of the sequence. The use of strategies based on recursive reasoning wasn't always made correctly, especially when far generalization questions were involved. The recursive approach through the use of D_2 lacked a final adjustment based on the context of the problem, because students only considered a multiple of the common difference, forgetting to add the last four pins or the last pin, depending on the interpretation. Also, when they used explicit strategies, the model wasn't always correctly applied. In some cases, students added pins and cards in the end. It0s possible that these errors are linked to the extensive experience of students in manipulating numbers without meaning, making no sense of what the coefficients in the linear pattern represent. The level of efficiency presented by students increased on the second task. They revealed more awareness on the selection of the proper strategies to use in each case, for example, the inadequate use of direct proportion is no longer observed. In spite of these differences, it was obvious that students experienced difficulties when reverse thinking is involved, being also clear that this type of questions provokes a shift on the type of approaches used by them.

The role of visualization in students' reasoning

According to Presmeg (1986) a strategy is considered visual if the image/drawing plays a central role in obtaining the answer, either directly or as a starting point for finding the rule. In this sense the following strategies are included in this group: *counting*, *whole-object with visual adjustment*, *difference with rate-adjustment* and *explicit*. *Counting* was always a successful strategy but only useful in solving near generalization questions. Drawing a picture of the object required and counting all the elements is an action used in near generalization questions and does not lead to a generalized strategy. Strategy W_3 was only used by one pair of students, when solving the first task. They've only applied it correctly in near generalization. This type of reasoning involves a higher level of abstraction in visualization, difficult to attain. In spite of not being one of the most frequent strategies, students who used D_3 always reached the correct answer. This fact reflects once more the relevance of understanding the context surrounding the problem, making the relation between variables clearer. Finally, the application of an *explicit* strategy lead to a high level of efficacy. Students based their work on the structure of the sequence, making reference to the relation between the variables reported on the problem. Only a few cases were registered that, along the way, disconnected from the context and mixed different variables.

Discussion

In this research, the main purpose of using pattern exploration tasks was setting an environment to analyse students' generalization strategies, difficulties emerging from their work, and the impact of using visual strategies in generalization.

As for the research questions outlined earlier in this paper, there are some pertinent observations: (a) a variety of strategies were identified in the work developed by students, although some were more frequent than others, like *counting* (mostly on near generalization) and *explicit* (more frequent on far generalization); (b) students achieved better results in near generalization questions than on far generalization questions and, even with some experience with patterning activities, reverse thinking was still complex for many of them; (c) some of the pairs worked exclusively on number contexts using inadequate strategies like the application of direct proportion, using multiples of the difference between two consecutive terms without a final adjustment and mixing variables. Along the study, this tendency was gradually inverted as most students understood the limitations of some of those strategies; (d) in some cases, students revealed difficulties finding a functional relation, frequently generalizing rules that verified for particular cases or showing a fixation for a recursive strategy; (e) visualization proved to be a useful ability in different situations like making a drawing and counting its elements, to solve near generalization tasks, and "seeing" the structure of the pattern, finding an *explicit* strategy to solve far generalization tasks; (f) the application of visual strategies allowed students to find different expressions to represent the same pattern.

To conclude, it's important to provide tasks which encourage students to use and understand the potential of visual strategies and to relate number context with visual context to better understand the meaning of numbers and variables. Establishing a clear connection between parallel approaches and exploring the potentialities and limitations of each case can contribute to the development of mathematical flexibility.

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